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ScienceDirect
Journal of Geometry and Physics 58 (2008) 238–252
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Review

Connections and the Dirac operator on spinor bundles☆

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Received 9 September 2007; accepted 8 November 2007
Available online 17 November 2007

Abstract

There are two approaches to spinor fields on a (pseudo-) Riemannian manifold (M, g): the bundle of spinors is either defined as a bundle associated with the principal bundle of 'spin frames' or as a complex bundle $\Sigma \to M$ with a homomorphism $\tau : \mathcal{C}\ell(g) \to \operatorname{End} \Sigma$ of bundles of algebras over M such that, for every $x \in M$, the restriction of τ to the fiber over x is equivalent to a spinor representation of a suitable Clifford algebra. By Hermitian and complex conjugation one obtains the homomorphisms $\tau^\dagger : \mathcal{C}\ell(g) \to \operatorname{End} \bar{\Sigma}^*$ and $\bar{\tau} : \mathcal{C}\ell(g) \to \operatorname{End} \bar{\Sigma}$. These data define the bundles $\mathfrak{a}(\tau)$ and $\mathfrak{c}(\tau)$ of intertwiners of τ with $\tau^\dagger$ and $\bar{\tau}$, respectively. It is shown that, given sections of $\mathfrak{a}(\tau) \to M$ and of $\mathfrak{c}(\tau) \to M$, any metric linear connection on (M, g) defines a unique connection on the spinor bundle $\Sigma \to M$ relative to which these sections are covariantly constant. The connection defines a Dirac operator acting on sections of Σ. As an example, the trivial spinor bundle on hypersurfaces in $\mathbb{R}^m$ and the corresponding Dirac operator are described in detail.

JGPSC: Spinors and twistors

MSC: 15A66; 53C7; 57R15; 81R25

PACS: 02.40.Ky; 04.20.Cv

Keywords: Spinor bundles; Covariant differentiation of spinor fields; Dirac operator

Contents

☆ Work on this article was supported in part by grant No. 1 P03B 075 29 of the Polish Ministry of Science and Higher Education.
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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.11.001

1. Introduction

Soon after the appearance of the Dirac equation [5], mathematicians and physicists extended it to curved, Lorentzian manifolds of Einstein's general relativity theory (GRT). When doing this, Vladimir A. Fock [6] and Hermann Weyl [31] introduced, on the curved manifold, orthonormal frames (Weyl: *Achsenkreuze*; Fock, following Einstein: *Beine*) and used a spinor connection and the constant Dirac matrices of special relativity to define the Dirac operator.

Another approach was initiated by Hugo Tetrode [26] and significantly developed by Erwin Schrödinger [22]: assuming local coordinates (x^μ) in a space–time with the line-element $\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$, they introduced point-dependent Dirac matrices $\gamma_\mu(x)$, satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}. \tag{1}$$

Schrödinger and, a little later, Infeld and van der Waerden [10], considered what nowadays is called a torsion-free spin^c connection, describing the interaction of charged fermions with gravitational and electromagnetic fields. Schrödinger derived a formula for the square of the Dirac operator that included a term involving the electromagnetic field, corresponding to the curvature of the U(1)-part of the connection. This Schrödinger–Lichnerowicz formula is an important tool in global analysis; it is now used in the theory of the Seiberg–Witten invariants; see section A.2 in [7] and the references given there. The electromagnetic potential appeared also in the spinor connection of the earlier work by Fock.

Both lines of approach used the notions of local differential geometry, prevalent at the time. Later, with the development of algebraic topology and global differential geometry, proper, intrinsic definitions of spin structures, spinor fields and the Dirac operator on manifolds have been given; see [14] and the bibliography given there.

The Schrödinger approach received little attention from mathematicians. Marcel Riesz [20,21] developed a theory of spinors based on the notion of minimal, one-sided ideals of Clifford algebras. Riesz, moreover, considered a 'local' Clifford algebra $\mathcal{C}\ell(g_x)$, associated with the tangent quadratic space $(T_x M, g_x)$ at a point x of a Lorentz manifold (M, g). Later, Guido Karrer [11] recognized that the set of all such local Clifford algebras can be given the structure of a Clifford bundle $\mathcal{C}\ell(g)$ over M. Spinor fields are then defined as sections of a vector bundle $\Sigma \to M$ carrying a representation of the Clifford bundle. Karrer proved the existence of a covariant derivative on Σ, compatible with the Levi-Civita connection on (M, g) [12]. He did not, however, consider the question of its uniqueness.

In the past, there has been much discussion about the relative advantages of the two approaches and the relations between them. It suffices to recall the sharp criticism by Élie Cartan of the work of Infeld and van der Waerden; he dismissed those authors as 'certain physicists' (see p. 150 in the English edition of [3]). Schrödinger was somewhat milder: referring to the approach based on *Achsenkreuze*, he wrote: *Bei diesem Verfahren ist es ein bißchen schwer zu erkennen, ob die Einsteinsche Idee des Fernparallelismus, auf die teilweise direkt Bezug genommen wird, wirklich hereinspielt oder ob man davon unabhängig ist*. ('With this approach, it is a little hard to see whether Einstein's idea of teleparallelism, to which reference is partially made, really is a related concept or whether one is independent of it.' Translation courtesy of Ilka Agricola). The arguments were often obscured by complicated notations and a lack of precise definitions of the underlying structures.

These *two approaches to spinor fields on Riemannian manifolds* can be now precisely formulated and compared using the notion of fiber bundles. Essentially,

(i) the first approach corresponds to using a *spin structure* defined in terms of a *principal fiber bundle* which is a 'reduction' of the bundle of orthonormal frames to the spin group;
(ii) the second is based on *spinor bundles* defined as *vector bundles* whose fibers carry spinor representations of the Clifford algebras $\mathcal{C}\ell(g_x)$; spinor fields are sections of spinor bundles.

One goes from (i) to (ii) by forming an *associated* bundle, but passing from a spinor bundle to the principal spin bundle is subtle and not always possible. There are essential differences between the relation of (ii) to (i) in even- and odd-dimensional manifolds [8]. There are, in both approaches, *topological obstacles* to the global existence of the structures required to describe spinors on manifolds.

Physicists sometimes say that they see no use for principal bundles: fields and wave-functions they consider are defined as sections of vector bundles. In GRT, two-component spinors have been introduced as an important tool by Roger Penrose [19]. It is worth to note that Penrose takes, as a starting point of his considerations, point-dependent Pauli matrices. In other words, he follows the second approach, as described above. He is not, however, entirely explicit about this matter.

This article extends the paper by Friedrich and Trautman [8] on the relation between the two approaches to spinors on manifolds by presenting a detailed description of connections on spinor bundles. The definition of connections on the principal bundles of spin structures is well known [7,14], and there is no need to recall it here. Throughout this article, it is assumed that the topological conditions, necessary for the existence of the structures under consideration, are satisfied.

2. A historical aside

Starting in 1935, there was a considerable amount of work on spinor fields, and their relations to geometry, done at the Physical Institute of Hiroshima University. Yositaka Mimura outlined a program to develop a 'wave geometry', intended to connect relativity with quantum mechanics. The starting point was the observation that (1) implies $\mathrm{d}s^2 = (\gamma_\mu \mathrm{d}x^\mu)^2$, an equation that can be formally 'solved' as $\mathrm{d}s = \gamma_\mu \mathrm{d}x^\mu$. Introducing a spinor field ψ describing the 'state of the four-dimensional space–time', Mimura wrote the basic equation of wave geometry as $\mathrm{d}s\psi = \gamma_\mu \mathrm{d}x^\mu \psi$. As a by-product of this research, there was the discovery that, in a complex space–time, the self-duality of the curvature tensor is the integrability condition of the equation $\nabla_\mu \psi = 0$, where ψ is a chiral spinor field. A general method of solving the self-duality condition has been outlined; see [17,24] and the reviews [15,16]. Strangely enough, those interesting early results are hardly ever mentioned in the present literature.

The nuclear attack on Hiroshima destroyed the building of the Institute and killed several members of its staff. In 1948, the Research Institute for Theoretical Physics (RITP) of Hiroshima University was moved to Takehara and, much later, incorporated into the Yukawa Institute in Kyoto.

In 1979, the author of this article visited RITP at Takehara and enjoyed there a graceful hospitality extended to him by Hyôitirô Takeno, one of the foremost members of the Hiroshima school.

3. Preliminaries

3.1. Notation and terminology

The notation and terminology used in this article closely follows that of [8]; see also [14,23]. Some of it is recalled here to make the paper self-contained.

If S and S' are finite-dimensional complex vector spaces, then the vector space of all complex linear maps of S into S' is denoted by $\mathrm{Hom}(S, S')$. The vector space $\mathrm{End}\, S = \mathrm{Hom}(S, S)$ is an algebra over $\mathbb{C}$ with respect to composition of endomorphisms; its unit is $I = \mathrm{id}_S$. One writes $S^* = \mathrm{Hom}(S, \mathbb{C})$; the *evaluation map* $S \times S^* \to \mathbb{C}$ is $(s, t) \mapsto \langle s, t\rangle$. If $f \in \mathrm{Hom}(S, S')$, then the *transpose* $f^* \in \mathrm{Hom}(S'^*, S^*)$ is defined by $\langle s, f^*(t)\rangle = \langle f(s), t\rangle$ for every $s \in S$ and $t \in S'^*$. Let S and S' be complex vector spaces. A map $f : S \to S'$ such that $f(\lambda s_1 + s_2) = \bar{\lambda} f(s_1) + f(s_2)$ for every $s_1, s_2 \in S$ and $\lambda \in \mathbb{C}$ is said to be *semi-linear* (sometimes: antilinear). The *complex conjugate* of the finite-dimensional complex vector space S is the complex vector space

$$\bar{S} = \{u : S^* \to \mathbb{C} \mid u \text{ is semi-linear}\}.$$

The map $S \to \bar{S}$, $s \mapsto \bar{s}$, given, for every $t \in S^*$ by $\bar{s}(t) = \overline{t(s)}$, is bijective and semi-linear. The complex conjugate of $f : S \to S'$ is the linear map $\bar{f} : \bar{S} \to \bar{S}'$ given by $\bar{f}(\bar{s}) = \overline{f(s)}$. The spaces $(\bar{S})^*$ and $\overline{S^*}$ are identified and denoted as $\bar{S}^*$. The *Hermitian conjugate* of $f : S \to S'$ is the map $f^\dagger \stackrel{\text{def}}{=} \bar{f}^* : \bar{S}'^* \to \bar{S}^*$. If $f : S \to \bar{S}^*$ and $f^\dagger = f$, then f is a *Hermitian* map. If $\mathfrak{l}$ is a complex or real line, then $\mathfrak{l}^\times = \mathfrak{l} \setminus \{0\}$.

Every algebra under consideration here has a unit element; homomorphisms of algebras map units into units. If $\mathfrak{A}$ is an algebra, then its derivations form a vector space

$$\operatorname{Der}\mathfrak{A} = \{d \in \operatorname{End}\mathfrak{A} \mid d(xy) = (dx)y + x(dy),\ \forall x, y \in \mathfrak{A}\}.$$

The following fact is 'well known', but hard to find in textbooks on algebra:

Proposition 1. *Every derivation of the algebra* End S *is inner, i.e. if* $d \in \operatorname{Der}\operatorname{End} S$, *then there is* $a \in \operatorname{End} S$ *such that*

$$dx = ax - xa \quad \text{for every } x \in \operatorname{End} S.$$

Proof. Let $s \in S$ and $t^* \in S^*$ be such that $\langle s, t^*\rangle = 1$. Given $d \in \operatorname{Der}\operatorname{End} S$, define $a \in \operatorname{End} S$ by $a(s') = (d(s' \otimes t^*))(s)$ for every $s' \in S$. Then dx is as above. $\square$

This proof was communicated to the author by Peter Šemrl; it is based on the proof given in [13] for the infinite-dimensional case.

A real *quadratic space* is defined as a pair (V, h), where V is a finite-dimensional real vector space and h is a non-degenerate scalar product. If the signature of the quadratic form $v \mapsto h(v, v)$ is (k, l), $k + l = m$, then there is an *orthonormal frame* $(\epsilon_1, \ldots, \epsilon_m)$ in V such that, writing $h_{\mu\nu} = h(\epsilon_\mu, \epsilon_\nu)$, one has $h_{\mu\mu} = -1$ for $\mu = 1, \ldots, k$, $h_{\mu\mu} = 1$ for $\mu = k + 1, \ldots, m$ and, if $\mu \neq \nu$, then $h_{\mu\nu} = 0$. The 'contravariant' components $h^{\mu\nu}$ are defined by $h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu$. (Numerically, for orthonormal frames, $h^{\mu\nu} = h_{\mu\nu}$.) The scalar product h defines an isomorphism $\tilde{h} : V \to V^*$ such that $\langle v', \tilde{h}(v)\rangle = h(v, v')$ for all $v, v' \in V$. *Throughout this article it is assumed that* k *or* $l > 1$. This is to exclude quadratic spaces having the property that the connected component of the spin group does not contain -1. It is convenient to label with h the various groups and other algebraic structures associated with (V, h). In particular, SO(h) and Spin(h) are the special orthogonal and spin groups associated with (V, h), respectively. The vector space $\mathbb{R}^m$ has a 'standard', positive-definite scalar product h_0 and a 'canonical' orthonormal frame $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ such that $\epsilon_1 = (1, 0, \ldots, 0)$, $\epsilon_2 = (0, 1, \ldots, 0), \ldots, \epsilon_m = (0, 0, \ldots, 1)$. One writes SO$(m)$ and Spin(m) instead of SO(h_0) and Spin(h_0), etc.

Let $\mathbb{Z}_2 = \{1, -1\}$ and let $1 \to \mathbb{Z}_2 \to G$ be an exact sequence of homomorphisms of groups so that $-1 \in G$. The group G^c is defined as $(\mathrm{U}(1) \times G)/\mathbb{Z}_2$: its elements are equivalence classes $[(z, a)]$ such that $[(z, a)] = [(z', a')]$ if, and only if, either $(z', a') = (z, a)$ or $(z', a') = (-z, -a)$ for $z, z' \in \mathrm{U}(1)$ and $a, a' \in G$.

3.2. Conventions concerning manifolds and bundles

All manifolds and maps among them are assumed to be smooth. Manifolds are finite-dimensional and orientable, but not necessarily compact. $C(M)$ denotes the algebra of smooth, real-valued functions on the manifold M. If $f : M \to M'$ is a map of manifolds, then $Tf : TM \to TM'$ is the corresponding tangent (derived) map. A vector field X on M acts in $C(M)$ by derivation; if $f, f' \in C(M)$, then $X(ff') = X(f)f' + fX(f')$. A *Riemannian manifold* (M, g) is a connected manifold M with a metric tensor field g which is non-degenerate, but not necessarily definite; if it is, then (M, g) is said to be *proper* Riemannian. If $\pi : E \to M$ is a fiber bundle over a manifold M, then $E_x = \pi^{-1}(x) \subset E$ is the fiber over $x \in M$; in particular, $T_xM \subset TM$ is the tangent vector space to M at x. A quadratic space (V, h) is said to be the *model* of (M, g) if, for every $x \in M$, the quadratic spaces (V, h) and (T_xM, g_x), $g_x = g|T_xM$, are isometric. An orthonormal frame (e_μ) in (T_xM, g_x) is identified with the isometry $e : V \to T_xM$ such that $e(\epsilon_\mu) = e_\mu$, $\mu = 1, \ldots, m$. An orientable Riemannian manifold with a local model (V, h) has a principal O(h)-bundle of all orthonormal frames that has two connected components; an orientation on such a manifold singles out one of them, referred to as *the* principal bundle of orthonormal frames,

$$\mathrm{SO}(h) \to P \xrightarrow{\pi} M. \tag{2}$$

4. Clifford algebras and spinors

4.1. Clifford algebras

Clifford algebras and their representations are well described in the literature; see [2,4] and the references given in [29]. This section is intended only to establish some of the notation and terminology used in the rest of the article.

Let (V, h) be a quadratic space of dimension m. Its tensor algebra, $\mathcal{T}V = \oplus_{p=0}^{\infty} \otimes^p V$, contains a two-sided ideal $\mathcal{J}(h)$ generated by all elements of the form $v \otimes v - h(v, v)$, $v \in V$. The *Clifford algebra* $\mathcal{C}\ell(h)$ is a part of the exact sequence of homomorphisms of algebras

$$0 \to \mathcal{J}(h) \to \mathcal{T}V \xrightarrow{\kappa} \mathcal{C}\ell(h) \to 0.$$

If $a, a' \in \mathcal{T}V$, then one writes $\kappa(a \otimes a') = \kappa(a)\kappa(a')$. Since the canonical map κ, restricted to $\mathbb{R} \oplus V \subset \mathcal{T}V$ is injective, one identifies $\mathbb{R} \oplus V$ with its image in the Clifford algebra so that one can write now $v^2 = h(v, v)$ for every $v \in V \subset \mathcal{C}\ell(h)$. Every Clifford algebra has a unit element, is associative and of dimension 2^m over $\mathbb{R}$. If $V = \mathbb{R}^m$ and h is positive definite, then one writes $\mathcal{C}\ell(m)$ instead of $\mathcal{C}\ell(h)$.

The following facts are classical; see, e.g., Ch. I in [14]. Clifford algebras are *universal*: let (V, h) be as before and let $\mathfrak{A}$ be an algebra over $\mathbb{R}$, with the unit element $1_{\mathfrak{A}}$. Every *Clifford map*, defined as a linear map $r : V \to \mathfrak{A}$ such that, for every $v \in V$, $(r(v))^2 = h(v, v)1_{\mathfrak{A}}$, extends to a homomorphism $\mathcal{C}\ell(h) \to \mathfrak{A}$ of algebras.

In particular, if (V, h) and (V', h') are quadratic spaces, and $i : V \to V'$ is an isometry, i.e. a linear map such that $h'(i(v), i(v)) = h(v, v)$ for every $v \in V$, then there is a homomorphism of algebras $\mathcal{C}\ell(i) : \mathcal{C}\ell(h) \to \mathcal{C}\ell(h')$ extending the Clifford map $V \to \mathcal{C}\ell(h')$, $v \mapsto i(v)$. This defines $\mathcal{C}\ell$ as a covariant functor from the category of quadratic spaces to that of algebras.

The isometry $v \mapsto -v$ extends to the involutive automorphism α of $\mathcal{C}\ell(h)$ which defines its $\mathbb{Z}_2$-grading, $\mathcal{C}\ell(h) = \mathcal{C}\ell^0(h) \oplus \mathcal{C}\ell^1(h)$. The involutive antiautomorphism β of $\mathcal{C}\ell(h)$ is characterized by $\beta(aa') = \beta(a')\beta(a)$ for all $a, a' \in \mathcal{C}\ell(h)$ and $\beta(a) = a$ for $a \in \mathbb{R} \oplus V$.

Let h be of signature (k, l), $k + l = m$. If $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$ is an orthonormal frame in V, then the volume element $\eta = \epsilon_1\epsilon_2 \ldots \epsilon_m$ defines an *orientation* in V. For m even (resp., odd) η anticommutes (resp., commutes) with all elements of V and

$$\eta^2 = (-1)^{\frac{1}{2}(k-l)(k-l+1)}. \tag{3}$$

4.2. Spinor representations

The algebra $\mathcal{C}\ell(h)$ (resp., $\mathcal{C}\ell^0(h)$) is simple and central for m even (resp., odd). If $m = 2n$, (resp., $m = 2n + 1$), then $\mathcal{C}\ell(h)$ (resp., $\mathcal{C}\ell^0(h)$) has a unique, up to equivalence, irreducible and faithful *Dirac representation* γ (resp., *Pauli representation*) in a *complex* vector space S of dimension 2^n.

For m odd, the Pauli representation extends to two complex inequivalent representations of the full algebra $\mathcal{C}\ell(h)$ in S: if σ is one of them, then $\sigma \circ \alpha$ is the other. It is sometimes convenient to consider (the decomposable but faithful) *Cartan representation* $\sigma \oplus (\sigma \circ \alpha)$.

The letter ρ denotes a generic *spinor representation*, without reference to the parity of m: in every dimension $m = 2n$ or $2n + 1$ of V, there is an irreducible representation

$$\rho : \mathcal{C}\ell(h) \to \operatorname{End} S \tag{4}$$

in a complex vector space S of dimension 2^n. The representations ρ and $\rho \circ \alpha$ are equivalent or not, depending on whether m is even or odd. If $(\epsilon_1, \ldots, \epsilon_m)$ is an orthonormal frame in $V \subset \mathcal{C}\ell(h)$ and S is identified with $\mathbb{C}^{2^n}$, then the endomorphisms $\rho_\mu = \rho(\epsilon_\mu)$, $\mu = 1, \ldots, m$, become for m even (resp., odd) the Dirac matrices γ_μ (resp., Pauli matrices σ_μ). This convention reflects the traditional notation, used in physics, in dimension 4 (resp., 3).

The algebra $\operatorname{End} S$ is generated over $\mathbb{C}$ by $\rho(V) \subset \operatorname{End} S$ and an injective Clifford map $r : V \to \operatorname{End} S$ *induces* a spinor representation (4). The homomorphism of algebras ρ is injective on $\mathbb{R} \oplus V$. In fact, ρ is injective on the full algebra $\mathcal{C}\ell(h)$ unless $k - l + 1$ is divisible by 4.

The representation (4) extends, in a natural way, to a spinor representation ρ^c of the complexification of $\mathcal{C}\ell(h)$,

$$\rho^c : \mathbb{C} \otimes \mathcal{C}\ell(h) \to \operatorname{End} S, \quad \rho^c(a + \mathrm{i}b) = \rho(a) + \mathrm{i}\rho(b), \quad a, b \in \mathcal{C}\ell(h). \tag{5}$$

The superscript ^{c} appearing in (5) is often omitted. If $m = 2n$ (resp., $m = 2n + 1$), then the complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}\ell(h)$ (resp., $\mathbb{C} \otimes \mathcal{C}\ell^0(h)$) is isomorphic, as a complex algebra, to End S.

4.3. *The intertwiners*

By considering the relation between the representation (4) and its *dual*,

$$\check{\rho} : \mathcal{C}\ell(h) \to \operatorname{End} S^*, \qquad \check{\rho}(a) = \rho(\beta(a))^*,$$

one finds that, for $m = 2n$ or $2n + 1$, there is an equivalence of representations: $\rho \sim \check{\rho}$ for n even and $\rho \circ \alpha \sim \check{\rho}$ for n odd. Therefore, in every dimension, there is a complex line $\mathfrak{b}(\rho)$ consisting of all $\mathsf{B} \in \operatorname{Hom}(S, S^*)$ such that

$$\rho(v)^*\mathsf{B} = (-1)^n \mathsf{B}\rho(v) \quad \forall v \in V \text{ of dimension } 2n \text{ or } 2n + 1. \tag{6}$$

One shows that $\mathsf{B}^* = (-1)^{\frac{1}{2}n(n+1)}\mathsf{B}$ (see Section 101 in [3], where the letter C plays the role of the present paper's B).

The *complex conjugate* of (4) is the representation

$$\bar{\rho} : \mathcal{C}\ell(h) \to \operatorname{End} \bar{S}, \qquad \bar{\rho}(a) = \overline{\rho(a)}.$$

By considering the relation between ρ and $\bar{\rho}$, and taking (3) into account, one shows that, in signature (k, l), there is a complex line consisting of all $\mathsf{C} \in \operatorname{Hom}(S, \bar{S})$ such that

$$\overline{\rho(v)}\mathsf{C} = (-1)^{\frac{1}{2}(k-l)(k-l+1)}\mathsf{C}\rho(v) \quad \text{for every } v \in V. \tag{7}$$

If $\mathsf{C} \neq 0$, then it is invertible, and, replacing C by $\lambda\mathsf{C}$, $\lambda \in \mathbb{C}^\times$ and choosing λ appropriately, one can achieve

$$\bar{\mathsf{C}}\mathsf{C} = \begin{cases} I & \text{for } k - l \equiv 0, 1, 2, 7 \bmod 8 \\ -I & \text{for } k - l \equiv 3, 4, 5, 6 \bmod 8. \end{cases} \tag{8}$$

With every representation (4) one associates, in this manner, a *circle*

$$\mathfrak{c}(\rho) = \{\mathsf{C} \in \operatorname{Hom}(S, \bar{S}) \mid \mathsf{C} \text{ satisfies (7) and (8)}\}.$$

Let B and C satisfy (6) and (7), respectively, and consider the map

$$\mathsf{A} \overset{\text{def}}{=} \bar{\mathsf{B}}\mathsf{C} : S \to \bar{S}^*. \tag{9}$$

One easily checks that $\mathsf{A}^{-1}\mathsf{A}^\dagger \in \operatorname{End} S$ is in the commutant of $\rho(V)$ so that $\mathsf{A}^\dagger = z\mathsf{A}$, where $z \in \mathrm{U}(1)$. Therefore, by rescaling of B, one can make A to be a Hermitian map in the sense defined in Section 3, so that

$$(\bar{\mathsf{B}}\mathsf{C})^\dagger = \bar{\mathsf{B}}\mathsf{C}. \tag{10}$$

In this manner, with every spinor representation ρ and $\mathsf{C} \in \mathfrak{c}(\rho)$, one associates the *real line*

$$\mathfrak{b}(\rho, \mathsf{C}) = \{\mathsf{B} \in \operatorname{Hom}(S, S^*) \mid \mathsf{B} \text{ satisfies (6) and (10)}\}.$$

Using (6), (7) and (9) and the congruence

$$k \equiv n + \frac{1}{2}(k - l)(k - l + 1) \bmod 2$$

valid for both $k + l = 2n$ and $k + l = 2n + 1$, one shows that A satisfies

$$\mathsf{A}\rho(v) = (-1)^k \rho(v)^\dagger \mathsf{A} \quad \text{for every } v \in V \tag{11}$$

and there is the real line,

$$\mathfrak{a}(\rho) = \{\mathsf{A} \in \mathrm{Hom}(S, \bar{S}^*) \mid \mathsf{A} \text{ satisfies (11) and } \mathsf{A} = \mathsf{A}^\dagger\}. \tag{12}$$

The treatment of complex conjugation of spin spaces, uniform with respect to the parity of m, was suggested to the author by Helmuth Urbantke. For Minkowski space, (7) reduces to the corresponding formula given in Appendix C to [23]. The present article's use of the letters A, B and C can be traced to Wolfgang Pauli [18].

The sesquilinear form

$$(s, s') \mapsto \langle \bar{s}, \mathsf{A}(s') \rangle \tag{13}$$

plays a fundamental role in the construction of real tensors from pairs of spinors. The letter B reminds one of the possibility of forming the *bilinear* form $(s, s') \mapsto \langle s, \mathsf{B}(s') \rangle$ and C is associated with *charge conjugation* of spinors, $s \mapsto \mathsf{C}^{-1}(\bar{s})$.

4.4. Example: Minkowski space

Let S be a two-dimensional complex vector space with the antisymmetric map $\varepsilon : S \to S^*$ introduced, in this context, by van der Waerden [30]. For every $v \in \mathrm{Hom}(S, \bar{S})$ one has $v^* \bar{\varepsilon} v = \varepsilon \det v$. The quadratic form $v \mapsto \det v$, restricted to the real, four-dimensional vector space

$$V = \{v \in \mathrm{Hom}(S, \bar{S}) \mid \bar{\varepsilon} v \text{ is Hermitian}\}$$

has signature (3, 1). The map $\gamma : V \to \mathrm{End}(S \oplus \bar{S})$ given by

$$\gamma(v) = \begin{pmatrix} 0 & -\bar{v} \\ v & 0 \end{pmatrix} \quad \text{satisfies } \gamma(v)^2 = (\det v)\, \mathrm{id}_{S \oplus \bar{S}}$$

and defines a Dirac representation of $\mathcal{C}\ell(\det)$. One sees by inspection that a possible choice of the intertwiners is

$$\mathsf{A} = \begin{pmatrix} 0 & \bar{\varepsilon} \\ \varepsilon & 0 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon} \end{pmatrix} \quad \text{and} \quad \mathsf{C} = \begin{pmatrix} 0 & \mathrm{id}_{\bar{S}} \\ \mathrm{id}_S & 0 \end{pmatrix}.$$

4.5. Spinor groups

A vector $v \in V \subset \mathcal{C}\ell(h)$ that is non-null, $h(v, v) \neq 0$, is invertible as an element of $\mathcal{C}\ell(h)$, $v^{-1} = v/h(v, v)$. Define the *Clifford group* $\Gamma(h)$ associated with the spinor representation (4) as the subset of $\mathrm{GL}(S)$ consisting of all elements of the form $\rho(v_1 v_2 \ldots v_{2p})$, where all the vs are non-null vectors and p is any positive integer.

The name 'Clifford group' and the use of Γ to denote that group, were introduced by Claude Chevalley [4]. André Weil pointed out that it was Rudolf Lipschitz, not Clifford, who introduced groups constructed out of 'Clifford numbers'. See [8] for references and further remarks on this subject.

To be precise, $\Gamma(h)$ is isomorphic to Chevalley's *special* Clifford group and there is the exact sequence

$$1 \to \mathbb{R}^\times \to \Gamma(h) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(h) \to 1, \tag{14}$$

where Ad is defined by

$$\rho(\mathrm{Ad}(a)v) = a\rho(v)a^{-1}. \tag{15}$$

Recall that the homomorphism Ad is surjective by virtue of the Cartan–Dieudonné theorem; see, e.g., Section 4 in [9].

Proposition 2. *Let η be a volume element. The group $\Gamma(h)^c = (\mathrm{U}(1) \times \Gamma(h))/\mathbb{Z}_2$ is isomorphic to the group* $\mathrm{Aut}(\rho)$ *of all orientation-preserving automorphisms of the spinor representation (4), i.e. to the group consisting of all $b \in \mathrm{GL}(S)$ such that $b\rho(V)b^{-1} = \rho(V)$ and $b\rho(\eta)b^{-1} = \rho(\eta)$.*

Proof. Clearly, $\Gamma(h)$ is a subgroup of $\mathrm{Aut}(\rho)$. Consider the homomorphism of groups

$$f : \Gamma(h)^c \to \mathrm{Aut}(\rho), \quad f([(z,a)]) = za.$$

f is surjective: if $b \in \mathrm{Aut}(\rho)$, then, from the exactness of (14), there is $a \in \Gamma(h)$ such that $\mathrm{Ad}(a) = \mathrm{Ad}(b)$. The element ba^{-1} is in the center of $\mathrm{End}\, S$ so that there is $w \in \mathbb{C}^\times$ such that $b = wa$ and then $f([(w/|w|, |w|a)]) = b$. The homomorphism f is injective because $U(1) \cap \Gamma(h) = \{1, -1\}$. □

Definition 1. A *spinor group* associated with the representation (4) is any closed subgroup G of $\mathrm{Aut}(\rho)$ such that $\mathrm{Ad}(G)$ contains the connected component $\mathrm{SO}_0(h)$ of the group $\mathrm{SO}(h)$.

Clearly, $\mathrm{Aut}(\rho)$ is the 'largest' spinor group associated with (4). Every spinor group is isomorphic to a subgroup of the group of invertible elements of $\mathbb{C} \otimes \mathcal{C}\ell^0(h)$. It is convenient, however, for the purposes of this article, to consider all spinor groups as subgroups of $\mathrm{GL}(S)$. From now on the group $\mathrm{Aut}(\rho)$ is identified with $\Gamma(h)^c \subset \mathrm{GL}(S)$. The 'vector' representation $\mathrm{Ad} : G \to \mathrm{SO}(h)$ is defined as in (15). Every spinor group has a spinor representation in S obtained by the evaluation of $a \in G \subset \mathrm{GL}(S)$ in S. The spin group

$$\mathrm{Spin}(h) = \{\rho(v_1 v_2 \dots v_{2p}) \mid \text{all } v\text{s are unit vectors}, \ p = 1, 2 \dots\}$$

and its connected component $\mathrm{Spin}_0(h)$ are spinor groups. There is the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(h) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(h) \to 1$$

and a similar sequence for the connected component.

Recall that a necessary and sufficient condition for $\rho(v_1 v_2 \dots v_{2p})$ to be in $\mathrm{Spin}_0(h)$ is that among the unit vectors $v_1, v_2, \dots, v_{2p}$ there be precisely an even number of vectors with negative squares.

Proposition 3. (i) *The action of the group $\Gamma(h)^c$ on $\mathfrak{a}(\rho)^\times$ given by $\mathsf{A} \mapsto a^\dagger \mathsf{A} a$ for $a \in \Gamma(h)^c$ and $\mathsf{A} \in \mathfrak{a}(\rho)^\times$ is transitive. Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers. Defining $\mathrm{N}_{\mathfrak{a}} : \Gamma(h)^c \to \mathbb{R}^\times$ by*

$$\mathrm{N}_{\mathfrak{a}}(a)\mathsf{A} = a^\dagger \mathsf{A} a \tag{16}$$

one obtains the exact sequence of homomorphisms of groups

$$1 \to \mathrm{Spin}_0(h)^c \to \Gamma(h)^c \xrightarrow{\mathrm{N}_{\mathfrak{a}}} \begin{cases} \mathbb{R}^\times \to 1 & \text{if } k \neq 0 \\ \mathbb{R}^+ \to 1 & \text{if } k = 0 \end{cases}$$

and $\mathrm{Spin}(h)^c = \mathrm{N}_{\mathfrak{a}}^{-1}(\{1, -1\})$. For every $s, s' \in S$ and $a \in \Gamma(h)^c$ one has

$$\langle \overline{a(s)}, \mathsf{A}(a(s')) \rangle = \mathrm{N}_{\mathfrak{a}}(a) \langle \bar{s}, \mathsf{A}(s') \rangle. \tag{17}$$

(ii) *The action of the group $\Gamma(h)$ on $\mathfrak{b}(\rho, \mathsf{C})^\times$ given by $\mathsf{B} \mapsto a^* \mathsf{B} a$ for $a \in \Gamma(h)$ and $\mathsf{B} \in \mathfrak{b}(\rho, \mathsf{C})^\times$ is transitive. Defining*

$$\mathrm{N}_{\mathfrak{b}}(a)\mathsf{B} = a^* \mathsf{B} a$$

one obtains the exact sequence of homomorphisms of groups

$$1 \to \mathrm{Spin}_0(h) \to \Gamma(h) \xrightarrow{\mathrm{N}_{\mathfrak{b}}} \begin{cases} \mathbb{R}^\times \to 1 & \text{if } k \neq 0 \\ \mathbb{R}^+ \to 1 & \text{if } k = 0 \end{cases}$$

and $\mathrm{Spin}(h) = \mathrm{N}_{\mathfrak{b}}^{-1}(\{1, -1\})$. For every $s, s' \in S$ and $a \in \Gamma(h)$ one has

$$\langle a(s), \mathsf{B}(a(s')) \rangle = \mathrm{N}_{\mathfrak{b}}(a) \langle s, \mathsf{B}(s') \rangle.$$

(iii) *The action of the groups $\Gamma(h)^c$ and $\mathrm{Spin}_0(h)^c$ on $\mathfrak{c}(\rho)$ given by $\mathsf{C} \mapsto \bar{a}^{-1} \mathsf{C} a$ for $a \in \Gamma(h)^c$ and $a \in \mathrm{Spin}_0(h)^c$, respectively, and $\mathsf{C} \in \mathfrak{c}(\rho)$, is transitive. Defining*

$$\mathrm{N}_{\mathfrak{c}}(a)\mathsf{C} = \bar{a}^{-1} \mathsf{C} a$$

one obtains the exact sequences of homomorphisms of groups

$$1 \to \Gamma(h) \to \Gamma(h)^c \xrightarrow{\mathrm{N}_{\mathfrak{c}}} \mathrm{U}(1) \to 1$$

and

$$1 \to \mathrm{Spin}_0(h) \to \mathrm{Spin}(h)^c \xrightarrow{\mathrm{N}_\mathfrak{c}} \mathrm{U}(1) \to 1,$$

respectively.

Proof. (i) One easily checks, from the definition (12), that if $\mathsf{A} \in \mathfrak{a}(\rho)^\times$ and $a \in \Gamma(h)^c$, then $a^\dagger \mathsf{A} a \in \mathfrak{a}(\rho)^\times$. Since $\mathfrak{a}(\rho)$ is a real line, there is $\mathrm{N}_\mathfrak{a}(a) \in \mathbb{R}^\times$ such that (16) holds. The map $\mathrm{N}_\mathfrak{a}$ is a homomorphism and does not depend on $\mathsf{A} \in \mathfrak{a}(\rho)^\times$. If $\lambda \in \mathbb{C}^\times$, then $\mathrm{N}_\mathfrak{a}(\lambda) = |\lambda|^2$. If $a = \rho(v_1 v_2 \ldots v_{2p}) \in \Gamma(h)$, then $\mathrm{N}_\mathfrak{a}(a) = v_1^2 v_2^2 \ldots v_{2p}^2$, so that if $k = 0$, then $\mathrm{N}_\mathfrak{a}$ is onto $\mathbb{R}^+$; otherwise it is onto $\mathbb{R}^\times$. If $a = \rho(v_1 v_2 \ldots v_{2p})$ is in the kernel of $\mathrm{N}_\mathfrak{a}$, then one can make all the vs to be unit vectors and, among them, there is precisely an even number of vectors with negative squares. The equivariance property (17) of the sesquilinear form (13) follows directly from (16). The proofs of parts (ii) and (iii) follow the same pattern. $\square$

5. Spinor structures

Let (M, g) be an oriented Riemannian manifold with (V, h) as its local model. Consider the bundle of orthonormal frames (2) and a spinor representation (4). Given a spinor group G such that $\mathrm{Ad} : G \to \mathrm{SO}(h)$ is surjective, one defines a *spinor G-structure* on M to be a reduction Q of P to the group G: it is given by the diagram of maps

$$\begin{array}{ccccc} G & \longrightarrow & Q & & \\ {\scriptstyle \mathrm{Ad}}\downarrow & & \downarrow{\scriptstyle \chi} & & \\ \mathrm{SO}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array} \tag{18}$$

such that, for every $a \in G$ and $q \in Q$ one has $\chi(qa) = \chi(q)\,\mathrm{Ad}(a)$. Another spinor G-structure on M, given by $Q' \xrightarrow{\chi'} P$, is *equivalent* to (18) if there is a diffeomorphism $f : Q \to Q'$ such that, for every $q \in Q$ and $a \in G$, one has $f(qa) = f(q)a$ and $\chi' \circ f = \chi$.

If the manifold (M, g) is *space-* and *time-orientable*, then its bundle of orthonormal frames can be reduced to $\mathrm{SO}_0(h)$. If G_0 is a spinor group such that $\mathrm{Ad}(G_0) = \mathrm{SO}_0(h)$, then one can consider a spinor G_0-structure defined as a reduction Q_0 of the $\mathrm{SO}_0(h)$-bundle P_0 to the group G_0, given by a diagram analogous to (18).

In standard terminology, one refers to a spinor structure by the name of the group G or G_0 under consideration. Thus, for example, if $G = \Gamma(h)$, $\mathrm{Spin}(h)^c$ or $\mathrm{Spin}(h)$, then one refers to (18) as describing a Clifford, spin^c or spin structure, respectively. If $f : G \to G'$ is an injective homomorphism of spin groups and there is a spinor G-structure Q, then one can extend (weaken) it by forming a spinor G'-structure $Q' = (Q \times G')/G$. In this sense, the Clifford^c structure is the weakest among them.

If an oriented (proper) Riemannian m-manifold M has teleparallelism, i.e. if its bundle of orthonormal frames is isomorphic to $M \times \mathrm{SO}(m)$, then it has a *trivial* spin structure such that $Q = M \times \mathrm{Spin}(m)$, but, unless $\mathrm{H}(M, \mathbb{Z}_2) = 0$, it has also non-trivial spin structures (see Ch. II section 1 in [14]).

6. Clifford and spinor bundles

6.1. Definitions

Let (M, g) be a Riemannian m-manifold with the real quadratic space (V, h) as its local model. The set

$$\mathcal{C}\ell(g) = \bigcup_{x \in M} \mathcal{C}\ell(g_x)$$

has a canonically defined structure of a bundle of algebras over M; the typical fiber of this *Clifford bundle* is $\mathcal{C}\ell(h)$. As a vector bundle, the Clifford bundle is isomorphic to the *Grassmann bundle* $\wedge TM = \oplus_{p=0}^m \wedge^p TM$. There is the even Clifford subbundle $\mathcal{C}\ell^0(g)$ of $\mathcal{C}\ell(g)$.

Definition 2. A *spinor bundle* for the Riemannian manifold (M, g) is a complex vector bundle $\Sigma \to M$ with a morphism

$$\tau : \mathcal{C}\ell(g) \to \mathrm{End}\,\Sigma \tag{19}$$

of bundles of algebras over M such that, for every $x \in M$, the restriction τ_x of τ to the Clifford algebra $\mathcal{C}\ell(g_x)$ is a spinor representation.

From the universality of Clifford algebras it follows that, to define (19), it is enough to give the restriction of τ to $TM \subset \mathcal{C}\ell(g)$, this restriction being subject to $\tau(v)^2 = g(v, v)\,\mathrm{id}_{\Sigma_x}$ for every $v \in T_x M$.

6.2. Relations between spinor bundles and spinor structures

Proposition 4. *With every Clifford* ^{c} *structure*

$$\begin{array}{ccccc} \Gamma(h)^c & \longrightarrow & Q^c & & \\ {\scriptstyle \mathrm{Ad}}\big\downarrow & & \big\downarrow{\scriptstyle \chi} & & \\ \mathrm{SO}(h) & \longrightarrow & P & \xrightarrow{\pi} & M, \end{array}$$

there is a canonically associated spinor bundle (19).

Proof. The proof is by construction: let ρ^c be the spinor representation (5); define the associated bundle

$$\Sigma = (Q^c \times S)/\Gamma(h)^c$$

so that an element of Σ is an equivalence class, $[(q, s)]$, $(q, s) \in Q^c \times S$, and $[(q', s')] = [(q, s)]$ if, and only if, there is $a \in \Gamma(h)^c$ such that $q' = qa$ and $s' = a^{-1}s$. Let $v \in T_x M$ and $q \in Q^c_x$ so that the orthonormal frame $\chi(q)$ is an isometry $V \to T_x M$. The morphism (19) is now defined by $\tau(v)[(q, s)] = [(q, \rho^c(\chi(q)^{-1}(v))s)]$. $\square$

For every $x \in M$, the spinor representation $\tau_x : \mathcal{C}\ell(g_x) \to \mathrm{End}\,\Sigma_x$ defines the real line $\mathfrak{a}(\tau_x)$ and the circle $\mathfrak{c}(\tau_x)$. The set

$$\mathfrak{a}(\tau) = \bigcup_{x \in M} \mathfrak{a}(\tau_x) \subset \mathrm{Hom}(\Sigma, \bar{\Sigma}^*) = \Sigma^* \otimes \bar{\Sigma}^* \tag{20}$$

has the structure of a real line bundle over M. The set

$$\mathfrak{c}(\tau) = \bigcup_{x \in M} \mathfrak{c}(\tau_x) \subset \mathrm{Hom}(\Sigma, \bar{\Sigma}) = \Sigma^* \otimes \bar{\Sigma} \tag{21}$$

has the structure of a *bundle of circles* over M: it is a principal U(1)-bundle. If this bundle is trivial, i.e. if it has a (global) section $\mathcal{C} : M \to \mathfrak{c}(\tau)$, then one can define the real line bundle over M,

$$\mathfrak{b}(\tau, \mathcal{C}) = \bigcup_{x \in M} \mathfrak{b}(\tau_x, \mathcal{C}(x)) \subset \mathrm{Hom}(\Sigma, \Sigma^*) = \Sigma^* \otimes \Sigma^*. \tag{22}$$

Proposition 5. *Let (M, g) be an oriented Riemannian manifold with (V, h) as the local model.*

(i) *To every spinor bundle* (19) *there corresponds a Clifford* ^{c} *structure* (18) *such that the associated spinor bundle is isomorphic to* (19).
(ii) *This Clifford* ^{c} *structure can be reduced to a* spin_0 *structure if, and only if, the line bundle* (20) *is trivial.*
(iii) *The Clifford* ^{c} *structure can be reduced to a Clifford structure if, and only if, the bundle of circles* (21) *is trivial.*
(iv) *If the bundle of circles is trivial, then the resulting Clifford structure can be reduced to a* spin_0 *structure if, and only if, the real line bundle* (22) *is trivial.*

Proof. (i) Consider a spinor representation (4) and define

$$Q^c = \{q \in \mathrm{Hom}(S, \Sigma_x), x \in M \mid \forall v \in V, q\rho(v)q^{-1} \in \tau_x(T_x M)\}.$$

The group $\Gamma(h)^c \subset \mathrm{GL}(S)$ acts in Q^c by composition, $Q^c \times \Gamma(h)^c \to Q^c$, $(q, a) \mapsto qa$. This action is free and transitive on the fibers of $Q^c \to M$, $q \mapsto x$. The isometry $V \to T_x M$, $v \mapsto \tau_x^{-1}(q\rho(v)q^{-1})$ gives an orthonormal frame $\chi(q)$ at x. This defines the projection $\chi : Q^c \to P$ such that $\chi(qa) = \chi(q)\,\mathrm{Ad}(a)$.

(ii) Let $\mathsf{A} \in \mathfrak{a}(\rho)$. If there is a section $\mathcal{A} : M \to \mathfrak{a}(\tau)$ of the line bundle (20), then the reduction Q_0 of Q^c to $\mathrm{Spin}_0(h)$ is

$$Q_0 = \{q \in Q^c \mid \bar{q}^* \mathcal{A}(x) q = \mathsf{A}\}.$$

(iii) Let $\mathsf{C} \in \mathfrak{c}(\rho)$. If there is a section $\mathcal{C} : M \to \mathfrak{c}(\tau)$ of the circle bundle, then the reduction Q of Q^c to $\Gamma(h)$ is

$$Q = \{q \in Q^c \mid \mathcal{C}(x) q = \bar{q}\mathsf{C},\ x \in M\}.$$

(iv) Let $\mathsf{B} \in \mathfrak{b}(\rho)$. If there is a section $\mathcal{B} : M \to \mathfrak{b}(\tau, \mathcal{C})$ of the line bundle (22), then the reduction Q_0 of Q to $\mathrm{Spin}_0(h)$ is

$$Q_0 = \{q \in Q \mid q^* \mathcal{B}(x) q = \mathsf{B}\}. \quad \square$$

7. Covariant differentiation on vector bundles

Recall (see, e.g., vol. II of [25]) that a *covariant derivative* ∇ on a real or complex vector bundle $\Sigma \to M$ is a bilinear map

$$\mathrm{Sec}\, TM \times \mathrm{Sec}\, \Sigma \to \mathrm{Sec}\, \Sigma, \qquad (X, \varphi) \mapsto \nabla_X \varphi$$

such that, for every $f \in C(M)$, one has

$$\nabla_X (f\varphi) = X(f)\varphi + f\nabla_X \varphi \quad \text{and} \quad \nabla_{fX}\varphi = f\nabla_X \varphi.$$

If ∇ and ∇' are two covariant derivatives on Σ, then there is a one-form ω on M, with values in $\mathrm{End}\, \Sigma$, such that

$$\nabla'_X \varphi = \nabla_X \varphi + \omega(X)\varphi. \tag{23}$$

(More precisely, $\omega : TM \to \mathrm{End}\, \Sigma$ is a morphism of vector bundles over M.) The covariant derivative ∇^* on the dual bundle Σ^* is defined by the Leibniz rule: if $\varphi \in \mathrm{Sec}\, \Sigma$ and $\psi \in \mathrm{Sec}\, \Sigma^*$, then

$$X(\langle \varphi, \psi \rangle) = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla^*_X \psi \rangle$$

and (23) implies

$$\nabla'^*_X \psi = \nabla^*_X \psi - \omega(X)^* \psi.$$

The covariant derivative $\bar{\nabla}$ on $\bar{\Sigma}$ is defined by $\bar{\nabla}_X \bar{\varphi} = \overline{\nabla_X \varphi}$.

The covariant derivatives on two vector bundles over M extend, in a natural manner, to tensor products of these bundles. In particular, the covariant derivative $\widetilde{\nabla}$ on $\mathrm{End}\, \Sigma$, induced by ∇, is given by

$$(\widetilde{\nabla}_X \Phi)(\varphi) = \nabla_X (\Phi(\varphi)) - \Phi(\nabla_X \varphi), \tag{24}$$

where $\Phi \in \mathrm{Sec}\,\mathrm{End}\, \Sigma$.

Changes of connections on Σ induce changes of the connections on the tensor products. In particular, (23) leads to

$$\widetilde{\nabla}'_X \Phi = \widetilde{\nabla}_X \Phi + [\omega(X), \Phi] \tag{25}$$

and, if ∇^a is the extension of ∇ to $\mathrm{Hom}(\Sigma, \bar{\Sigma}^*)$, then

$$\nabla'^a_X \mathcal{A} = \nabla^a_X \mathcal{A} - \mathcal{A}\omega(X) - \bar{\omega}(X)^* \mathcal{A} \quad \text{for } \mathcal{A} \in \mathrm{Sec}\,\mathrm{Hom}(\Sigma, \bar{\Sigma}^*). \tag{26}$$

Proposition 6. *Let ∇ and ∇' be two covariant derivatives on a complex vector bundle $\Sigma \to M$ such that $\widetilde{\nabla} = \widetilde{\nabla}'$. Then there exists a $\mathbb{C}$-valued one-form α on M such that*

$$\nabla'_X = \nabla_X + \alpha(X)\, \mathrm{id}_{\mathrm{Sec}\, \Sigma}\,. \tag{27}$$

Proof. Indeed, using (25) one obtains

$$\omega(X)\Phi - \Phi\omega(X) = 0 \quad \text{for every } \Phi \in \operatorname{Sec}\operatorname{End}\Sigma$$

and the conclusion follows from Schur's Lemma. □

Let $\mathfrak{A} \to M$ be a bundle of algebras. A covariant derivative ∇ on the vector bundle $\mathfrak{A} \to M$ is said to be *adapted* to the algebra structure of this bundle if the map

$$\operatorname{Sec}\mathfrak{A} \to \operatorname{Sec}\mathfrak{A}, \quad \Phi \mapsto \nabla_X \Phi$$

is a derivation of the algebra $\operatorname{Sec}\mathfrak{A}$ for every $X \in \operatorname{Sec} TM$, i.e. if

$$\nabla_X(\Phi\Phi') = (\nabla_X \Phi)\Phi' + \Phi\nabla_X\Phi' \quad \text{for all } \Phi, \Phi' \in \operatorname{Sec}\mathfrak{A} \text{ and } X \in \operatorname{Sec} TM.$$

For example, the covariant derivative $\widetilde{\nabla}$, defined by Eq. (24), is adapted to the algebra structure of the bundle $\operatorname{End}\Sigma$.

Let ∇^{TM} be a covariant derivative on $TM \to M$; it extends, in a natural way, to a covariant derivative $\nabla^{\mathcal{T}M}$ on the tensor algebra $\mathcal{T}M$. The derivative $\nabla^{\mathcal{T}M}$ is adapted to the algebra structure of the tensor algebra.

Proposition 7. *Let (M, g) be a Riemannian manifold. The covariant derivative $\nabla^{\mathcal{T}M}$ on $\mathcal{T}M$ descends to a covariant derivative $\nabla^{\mathcal{C}\ell(g)}$ on $\mathcal{C}\ell(g)$, adapted to the algebra structure of $\mathcal{C}\ell(g)$ if, and only if, the connection ∇ is metric, $\nabla g = 0$.*

Proof. Indeed, let $X, Y \in \operatorname{Sec} TM$; the section $Y \otimes Y - g(Y, Y)$ of the ideal $\mathcal{I}(M, g) \subset \mathcal{T}M$ is mapped to 0 by the canonical map κ. The covariant derivative $\nabla_X^{\mathcal{T}M}(Y \otimes Y - g(Y, Y))$ is a section of the same ideal if, and only if, $(\nabla_X g)(Y, Y) = 0$. □

The covariant derivative $\nabla^{\mathcal{C}\ell(g)}$ on $\mathcal{C}\ell(g)$ restricts to a covariant derivative on the even Clifford bundle $\mathcal{C}\ell^0(g)$ and extends to complexifications of these bundles.

If $\Phi \in \operatorname{Sec}\operatorname{End}\Sigma$, then the transposed endomorphism $\Phi^* \in \operatorname{Sec}\operatorname{End}\Sigma^*$ is defined by

$$\langle \varphi, \Phi^*\psi \rangle = \langle \Phi\varphi, \psi \rangle \tag{28}$$

for every $\varphi \in \operatorname{Sec}\Sigma$ and $\psi \in \operatorname{Sec}\Sigma^*$. Differentiating both sides of equation (28) in the direction of X, using (24) and a similar equation defining the covariant derivative $\widetilde{\nabla}^*$ induced on $\operatorname{End}\Sigma^*$, one obtains $\widetilde{\nabla}^*_X \Phi^* = (\widetilde{\nabla}_X \Phi)^*$.

Proposition 8. *Every covariant derivative $\nabla^{\operatorname{End}\Sigma}$ on $\operatorname{End}\Sigma$, adapted to the algebra structure of $\operatorname{End}\Sigma$, is induced by a covariant derivative ∇ on Σ.*

Proof (*Adapted from Section* 4.4 *of [12]*). Let $\operatorname{End}_0\Sigma$ be the subbundle of $\operatorname{End}\Sigma$ consisting of endomorphisms of Σ with vanishing trace. Let $\operatorname{Der}\Sigma$ be the bundle of derivations of the fibers of $\operatorname{End}\Sigma$. By Proposition 1, all derivations of the algebra of all square matrices are inner and there is an isomorphism of vector bundles

$$j : \operatorname{End}_0\Sigma \to \operatorname{Der}\Sigma$$

such that, for every $x \in M$, $t_0 \in \operatorname{End}_0\Sigma_x$ and $t \in \operatorname{End}\Sigma_x$, one has

$$j(t_0)t = t_0 t - t t_0. \tag{29}$$

Let $\nabla^{\operatorname{End}\Sigma}$ be a covariant derivative on $\operatorname{End}\Sigma$, which is a derivation of the algebra $\operatorname{Sec}\operatorname{End}\Sigma$, and let ∇ be any covariant derivative on Σ. The bundle map $\varpi : TM \to \operatorname{End}\operatorname{End}\Sigma$ defined as in (23) by the difference $\nabla^{\operatorname{End}\Sigma} - \widetilde{\nabla}$ has values in $\operatorname{Der}\Sigma$. Therefore, there exists a map $\omega : TM \to \operatorname{End}_0\Sigma$ such that $\varpi = j \circ \omega$ and

$$\nabla_X^{\operatorname{End}\Sigma} - \widetilde{\nabla}_X = j(\omega(X)).$$

In view of (29), for every $\Phi \in \operatorname{Sec}\operatorname{End}\Sigma$, one has

$$\nabla_X^{\operatorname{End}\Sigma}\Phi - \widetilde{\nabla}_X\Phi = \omega(X)\Phi - \Phi\omega(X).$$

Define a new connection ∇' on Σ by (23). Then $\widetilde{\nabla}' = \nabla^{\operatorname{End}\Sigma}$. □

8. Covariant differentiation on spinor bundles

Consider a Riemannian manifold (M, g) with a metric covariant derivative ∇^{TM} on the tangent bundle $TM \to M$. According to Proposition 7, ∇^{TM} induces a covariant derivative $\nabla^{\mathcal{C}\ell}$ on $\mathcal{C}\ell(g)$. Given a spinor bundle $\Sigma \to M$, one can use the morphism (19) to transfer $\nabla^{\mathcal{C}\ell}$ to the bundle End Σ. To every vector field $X : M \to TM \subset \mathcal{C}\ell(g)$ there corresponds the section $\tau(X)$ of the bundle End $\Sigma \to M$. The transferred connection $\nabla^{\mathrm{End}\,\Sigma}$ is adapted to the algebra structure of End Σ and is characterized by

$$\nabla_X^{\mathrm{End}\,\Sigma}\tau(Y) = \tau(\nabla_X^{TM} Y) \tag{30}$$

so that "τ is covariantly constant". According to Propositions 6 and 8, there is a connection ∇ on Σ such that

$$\widetilde{\nabla} = \nabla^{\mathrm{End}\,\Sigma}. \tag{31}$$

It is defined up to the replacement of ∇_X by ∇'_X, as in (27).

Proposition 9. *Consider a spinor bundle $\Sigma \to M$ with the covariant derivative $\nabla^{\mathrm{End}\,\Sigma}$ defined as in (30).*

(i) *If there exists a section $\mathcal{A}$ of the line bundle (20), then there is a covariant derivative ∇ on Σ such that (31) holds and $\mathcal{A}$ is covariantly constant with respect to ∇. If ∇' is another covariant derivative on Σ with the same properties, then the one-form α defined by (27) is pure imaginary, $\bar{\alpha} = -\alpha$.*

(ii) *If there exists a section $\mathcal{C}$ of the circle bundle (21), then there is a covariant derivative ∇ on Σ such that (31) holds and $\mathcal{C}$ is covariantly constant with respect to ∇. If ∇' is another covariant derivative on Σ with the same properties, then the one-form α defined by (27) is real.*

(iii) *If there exist sections $\mathcal{A}$ and $\mathcal{C}$ described in* (i) *and* (ii) *, then there is a unique covariant derivative ∇ on Σ such that (31) holds and both these sections are covariantly constant with respect to ∇.*

(iv) *If the assumptions of* (iii) *are satisfied, then the Dirac operator D acting on sections of $\Sigma \to M$ is globally defined as follows. Let U_ι be an open subset of M and let $e = (e_\mu)_{\mu=1,\ldots,m}$ be a field of (not necessarily orthonormal) frames on U_ι. For every $p \in U_\iota$, the components of the metric tensor g with respect to e at p are $g_{\mu\nu}(p) = g(e_\mu(p), e_\nu(p))$ and there is the inverse $g^{\mu\nu}(p)$ of $g_{\mu\nu}(p)$. The restriction of the Dirac operator to U_ι is*

$$D = g^{\mu\nu}\tau(e_\mu)\nabla_{e_\nu}. \tag{32}$$

The Dirac operator on M is well defined by its restrictions to the sets (U_ι) providing an open cover of M.

Proof. (i) According to (12), one has $\mathcal{A}^\dagger = \mathcal{A}$ and, for every $Y \in \mathrm{Sec}\,TM$, $\mathcal{A}\tau(Y) = (-1)^k\tau(Y)^\dagger\mathcal{A}$. Covariantly differentiating both equations in the direction of $X \in \mathrm{Sec}\,TM$, one obtains $\nabla_X\mathcal{A}^\dagger = \nabla_X\mathcal{A}$ and $\nabla_X\mathcal{A} = \lambda(X)\mathcal{A}$, where λ is a real-valued form. Since $\mathcal{A} \in \mathrm{Sec}(\Sigma \otimes \bar{\Sigma})$, if ∇' is as in (27), then $\nabla'_X\mathcal{A} = \nabla_X\mathcal{A} + \alpha(X)\mathcal{A} + \bar{\alpha}(X)\mathcal{A}$ so that to obtain $\nabla'_X\mathcal{A} = 0$ it suffices to take α such that $\alpha + \bar{\alpha} = -\lambda$.

(ii) The proof proceeds similarly as in (i). Since $\mathcal{C} \in \mathrm{Sec}\,\Sigma \otimes \bar{\Sigma}^*$, one has now $\nabla'_X\mathcal{C} = \nabla_X\mathcal{C} + \alpha(X)\mathcal{C} - \bar{\alpha}(X)\mathcal{C}$.

(iii) This is a simple corollary from (i) and (ii).

(iv) If e' is a field of frames on another open subset U' of M, then on $U \cap U'$ one has $D' = D$ so that the Dirac operator is globally defined. □

Note that the connection ∇^{TM} giving rise to the connection on the spinor bundle is required to be metric, but may have torsion. In fact, there are mathematical and physical reasons to consider metric connections with torsion in the context of spinors; see [1] for a review and references.

The pure imaginary form α appearing in part (i) of the Theorem has been interpreted, early on, as representing the vector potential of an electromagnetic field [6,10,31].

9. Spinors on orientable hypersurfaces in $\mathbb{R}^m$

Consider $\mathbb{R}^m$ as a (flat) Riemannian manifold with the Cartesian coordinates (x^i) and a positive-definite metric tensor $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ defined by the scalar product $(.|.)$ on $\mathbb{R}^m$ so that $h_{ij} = (\epsilon_i|\epsilon_j)$, where $\epsilon_i = \partial_i$ and $i, j = 1, \ldots, m$. Let M be an oriented hypersurface in $\mathbb{R}^m$ with a metric tensor g induced by immersion from that in

$\mathbb{R}^m$. For every $p \in M$, the tangent space T_pM is identified with a subspace of codimension 1 in $\mathbb{R}^m$. The orientation of M defines a field $N : M \to \mathbb{R}^m$ of unit normals, $(N|N) = 1$, so that if $X \in T_pM$, then $(X|N(p)) = 0$. Let $N_i = h_{ij}N^j$. The vector fields

$$N_{ij} = N_i\partial_j - N_j\partial_i, \quad 1 \leqslant i, j \leqslant m, \tag{33}$$

are tangent to M. The trace of the second fundamental form of the hypersurface M is

$$\operatorname{div} N = (h^{ij} - N^iN^j)\partial_i N_j. \tag{34}$$

Let $\overset{\circ}{\nabla}$ be the covariant derivative of the flat Riemannian connection on $\mathbb{R}^m$ so that $\overset{\circ}{\nabla}_{\epsilon_i} = \partial_i$. The induced Riemannian connection ∇^{TM} on (M, g) is given, for a vector field Y tangent to M and $X \in T_pM$, by

$$\nabla^{TM}_X Y = \overset{\circ}{\nabla}_X Y + (\overset{\circ}{\nabla}_X N|Y)N.$$

Let the vector space S of spinors be of complex dimension 2^n. For $m = 2n$ or $2n+1$, there is a spinor representation $\rho : \mathcal{C}\ell(m) \to \operatorname{End} S$ and the hypersurface M has a trivial spinor bundle $\Sigma = M \times S \to M$ [8]. The morphism (19) is obtained from $\tau(u)(p, s) = (p, \rho(u)s)$, where $u = u^i\epsilon_i \in T_pM \subset \mathbb{R}^m$, $s \in S$, $\rho(u) = u^i\rho_i$, and $\rho_i = \rho(\epsilon_i)$ so that

$$\rho_i\rho_j + \rho_j\rho_i = 2h_{ij}I.$$

The bundle of intertwiners $\mathfrak{a}(\tau) \to M$ is trivial, $\mathfrak{a}(\tau) = M \times \mathfrak{a}(\rho)$; similarly for $\mathfrak{b}(\tau)$ and $\mathfrak{c}(\tau) \to M$.

The spinor connection, associated with the Riemannian connection on M, is given by the covariant derivative of a spinor field $\varphi : M \to S$,

$$\nabla_X\varphi = \overset{\circ}{\nabla}_X\varphi + \omega(X)\varphi, \quad \text{where } \omega(X) = \frac{1}{4}\rho(N\overset{\circ}{\nabla}_X N - \overset{\circ}{\nabla}_X NN). \tag{35}$$

The vectors N and $\overset{\circ}{\nabla}_X N$ appearing in the definition of ω are considered as elements of $\mathcal{C}\ell(m)$. To prove (35), one first checks the Leibniz rule

$$\nabla_X(\rho(Y)\varphi) = \rho(\nabla^{TM}_X Y)\varphi + \rho(Y)\nabla_X\varphi$$

and then proceeds to consider the intertwiners: if $\mathsf{A} \in \mathfrak{a}(\rho)$, then the section $\mathcal{A} : M \to \mathfrak{a}(\tau)$, such that $\mathcal{A}(p) = (p, \mathsf{A})$ for every $p \in M$, is seen to be *covariantly* constant by virtue of (11) and (26). Similarly for the other intertwiners.

Proposition 10. *The Dirac operator on the hypersurface M in $\mathbb{R}^m$ is*

$$D = \frac{1}{2}\rho(N)(\rho^i\rho^jN_{ij} - I\operatorname{div} N), \tag{36}$$

where N_{ij} and $\operatorname{div} N$ *are as in (33) and (34).*

Proof. Let (e_μ), $\mu = 1, \ldots, m-1$, be a local field of frames on $U \subset M$, $e_\mu(p) \in T_pM$. Let $(g^{\mu\nu})$ be the inverse of $g_{\mu\nu} = (e_\mu|e_\nu)$ so that (32) can be written as $D = g^{\mu\nu}\rho(e_\mu)\nabla_{e_\nu}$. Defining $e_m = N$, one extends the field of frames (e_μ) to the field of frames (e_i), $i = 1, \ldots, m$. Writing $e_i = E_i^j\epsilon_j$, one has $E_m^i = N^i$, $h_{ij}N^iE_\mu^j = 0$ and

$$h^{ij} = g^{\mu\nu}E_\mu^iE_\nu^j + N^iN^j.$$

Therefore

$$\rho(N)g^{\mu\nu}\rho(e_\mu)\overset{\circ}{\nabla}_{e_\nu} = g^{\mu\nu}E_\mu^iE_\nu^jN^k\rho_k\rho_i\partial_j = \frac{1}{2}\rho^i\rho^jN_{ij}.$$

A similar calculation gives

$$\rho(N)g^{\mu\nu}\rho(e_\mu)\omega(e_\nu) = -\frac{1}{2}\, I\operatorname{div} N$$

thus completing the proof of (36). □

The formula (36) was obtained in [27] under the assumption that the bundle of spinors $\Sigma \to M$ and its connection are derived from a principal bundle defining the spin structure on the hypersurface M. It was applied to derive, in a simple manner, the spectrum of the Dirac operator on spheres [28].

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